# Markscheme

## May 2023

## Mathematics: analysis and approaches

## Higher level

## Paper 1

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## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***R*** Marks awarded for clear **Reasoning**.

***AG*** Answer given in the question and so no marks are awarded.

***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

### 1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1**, **A2**.*

### 2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

| | **Correct answer seen** | **Further working seen** | **Any FT issues?** | **Action** |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | $5.65685...$ *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

**3 Implied marks**

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

**4 Follow through marks (only applied after an error is made)**

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$ , non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

**5 Mis-read**

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

**6 Alternative methods**

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

**7 Alternative forms**

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and $1.000$.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

**8 Format and accuracy of answers**

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

**9 Calculators**

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

**10. Presentation of candidate work**

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

## Section A

**1.** (a) recognizing $f\left(x\right)=0$ ***(M1)***

$x=-1$ ***A1***

***[2 marks]***

(b) (i) $x=2$ (must be an equation with $x$) ***A1***

(ii) $y=\frac{7}{2}$ (must be an equation with $y$) ***A1***

***[2 marks]***

(c) **EITHER**

interchanging $x$ and $y$ ***(M1)***

$2xy-4x=7y+7$

correct working with y terms on the same side: $2xy-7y=4x+7$ ***(A1)***

**OR**

$2yx-4y=7x+7$

correct working with $x$ terms on the same side: $2yx-7x=4y+7$ ***(A1)***

interchanging $x$ and $y$ OR making $x$ the subject $x=\frac{4y+7}{2y-7}$ ***(M1)***

**THEN**

$f^{-1}(x)=\frac{4x+7}{2x-7}$ (or equivalent) $\left(x\neq\frac{7}{2}\right)$ ***A1***

***[3 marks]***

***Total [7 marks]***

**2.** (a) (i) summing frequencies of riders or finding complement ***(M1)***

probability $= \frac{34}{40}$ ***A1***

(ii) attempt to find expected value ***(M1)***

$$\frac{16}{40}+\left(2\times\frac{13}{40}\right)+\left(3\times\frac{2}{40}\right)+\left(4\times\frac{3}{40}\right)$$

$\frac{60}{40}(=1.5)$ ***A1***

***[4 marks]***

(b) evidence of **their** rides/visitor $\times 1000$ or $\div 10$ ***(M1)***

$1500$ OR $0.15$

$150$ (times) ***A1***

***[2 marks]***

***Total [6 marks]***

**3.** $1-2\sin^2 x=\sin x$ ***A1***

$2\sin^2 x+\sin x-1=0$

valid attempt to solve quadratic ***(M1)***

$(2\sin x-1)(\sin x+1)$ OR $\dfrac{-1\pm\sqrt{1-4(2)(-1)}}{2(2)}$

recognition to solve for $\sin x$ ***(M1)***

$\sin x=\dfrac{1}{2}$ OR $\sin x=-1$

any correct solution from $\sin x=-1$ ***A1***

any correct solution from $\sin x=\dfrac{1}{2}$ ***A1***

**Note:** The previous two marks may be awarded for degree or radian values, irrespective of domain.

$x=-\dfrac{\pi}{2},\dfrac{\pi}{6},\dfrac{5\pi}{6}$ ***A1***

**Note:** If no working shown, award no marks for a final value(s).
Award ***A0*** for $-\dfrac{\pi}{2},\dfrac{\pi}{6},\dfrac{5\pi}{6}$ if additional values also given.

***Total [6 marks]***

**4.** recognition of quadratic in $e^x$ ***(M1)***

$\left(e^x\right)^2 - 3e^x + \ln k\,(=0)$ OR $A^2 - 3A + \ln k\,(=0)$

recognizing discriminant ≥ 0 (seen anywhere) ***(M1)***

$\left(-3\right)^2 - 4(1)(\ln k)$ OR $9 - 4\ln k$ ***(A1)***

$\ln k \le \frac{9}{4}$ ***(A1)***

$e^{9/4}$ (seen anywhere) ***A1***

$0 < k \le e^{9/4}$ ***A1***

***[6 marks]***

**5.** (a) recognition that period is $4m$ OR substitution of a point on $f$ ( except the origin) ***(M1)***

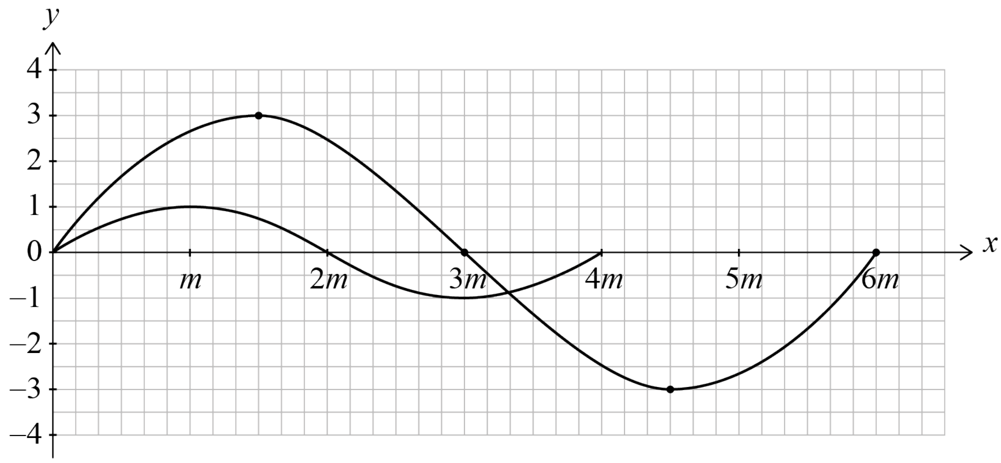
$4m = \dfrac{2\pi}{q}$ OR $1 = \sin qm$

$m = \dfrac{\pi}{2q}$ ***A1***

***[2 marks]***

(b) horizontal scale factor is $\dfrac{3}{2}$ (seen anywhere) ***(A1)***

**Note:** This ***(A1)*** may be earned by seeing a period of $6m$ , half period of $3m$ or the correct $x$-coordinate of the maximum/minimum point.




***A1A1A1***

**Note:** Curve must be an approximate sinusoidal shape (sine or cosine).
Only in this case, award the following:
***A1*** for correct amplitude.
***A1*** for correct domain.
***A1*** for correct max and min points **and** correct $x$-intercepts.

***[4 marks]***

***Total [6 marks]***

**6.** $A=\frac{1}{2}x^2\sin\frac{\pi}{3}$ OR $A=\frac{1}{2}x^2\sin 60^\circ$ OR triangle height $h=\sqrt{x^2-\left(\frac{x}{2}\right)^2}$ $\left(=\frac{\sqrt{3}}{2}x\right)$ ***(A1)***

$=\frac{1}{2}x^2\left(\frac{\sqrt{3}}{2}\right)$ OR $A=\frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right)\left(=\frac{\sqrt{3}}{4}x^2\right)$ ***A1***

**Note:** Award ***A1*** for $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$. This may be seen at a later stage.

attempt to use chain rule or implicit differentiation ***(M1)***

$\frac{dA}{dt}=\frac{dA}{dx}\times\frac{dx}{dt}$

$\frac{dA}{dt}=\frac{\sqrt{3}}{4}\times 2x\frac{dx}{dt}$ OR $\frac{dA}{dt}=\frac{1}{2}\times\sin\frac{\pi}{3}\times 2x\frac{dx}{dt}$ ***(A1)***

$=\frac{2\sqrt{3}}{4}\times 5\sqrt{3}\times 4$

$\frac{dA}{dt}=30\,(\text{cm}^2\text{s}^{-1})$ ***A1***

**Note:** Award a maximum of ***(A1)A1(M1)(A0)A1*** for a correct answer with incorrect derivative notation seen throughout.

***[5 marks]***

**7.** **METHOD 1**

$3\mathrm{i}$ (is a root) ***A1***

(other complex root is) $-3\mathrm{i}$ ***A1***

**Note:** Award ***A1A1*** for $P(3\mathrm{i})=0$ and $P(-3\mathrm{i})=0$ seen in their working.
Award ***A1*** for each correct root seen in sum or product of their roots.

**EITHER**

attempt to find $P(3\mathrm{i})=0$ or $P(-3\mathrm{i})=0$ ***(M1)***

$$4m-3m\mathrm{i}+\frac{36}{m}(3\mathrm{i})^2-(3\mathrm{i})^3=0$$

$$4m-3m\mathrm{i}-\frac{36}{m}(-9)+27\mathrm{i}=0$$

attempt to equate the real or imaginary parts ***(M1)***

$27-3m=0$ OR $9\times\frac{36}{m}=4m$

**OR**

attempt to equate sum of three roots to $\frac{36}{m}$ ***(M1)***

**Note:** Accept sum of three roots set to $-\frac{36}{m}$.
Award ***M0*** for stating sum of roots is $\pm\frac{36}{m}$.

$$3\mathrm{i}-3\mathrm{i}+r=\frac{36}{m}\left(\Rightarrow r=\frac{36}{m}\right)$$

substitute their $r$ into product of roots ***(M1)***

$(3\mathrm{i})(-3\mathrm{i})\left(\frac{36}{m}\right)=4m$ OR $(z^2+9)\left(\frac{36}{m}-z\right)$

$9\times\frac{36}{m}=4m$ OR $\frac{4m}{9}=\frac{36}{m}$

*continued…*

*Question 7 continued*

**OR**

attempt to equate product of three roots to $4m$ ***(M1)***

| |
|---|
| **Note:** Accept product of three roots set to $-4m$.<br>Award ***M0*** for stating product of roots is $\pm 4m$. |

$(3\text{i})(-3\text{i})\times r = 4m\left(\Rightarrow r = \frac{4m}{9}\right)$

substitute their $r$ into sum of roots ***(M1)***

$3\text{i} - 3\text{i} + \frac{4m}{9} = \frac{36}{m}$ OR $\left(z^2+9\right)\left(\frac{4m}{9} - z\right)$

$\frac{4m}{9} = \frac{36}{m}$

**THEN**

$m = 9$ ***(A1)***

third root is 4 ***A1***

***[6 marks]***

*Question 7 continued*

**METHOD 2**

$3\mathrm{i}$ (is a root) ***A1***

(other complex root is) $-3\mathrm{i}$ ***A1***

recognition that the other factor is $(z+3\mathrm{i})$ and attempt to write $P(z)$ as product of three linear factors or as product of a quadratic and a linear factor ***(M1)***

$P(z)=(z-3\mathrm{i})(z+3\mathrm{i})(r-z)$ OR $(z-3\mathrm{i})(z+3\mathrm{i})=z^2+9 \Rightarrow P(z)=(z^2+9)\left(\frac{4m}{9}-z\right)$

> **Note:** Accept any attempt at long division of $P(z)$ by $z^2+9$.
> Award ***M0*** for stating other factor is $(z+3\mathrm{i})$ or obtaining $z^2+9$ with no further working.

attempt to compare their coefficients ***(M1)***

$-9=-m$ OR $\frac{4m}{9}=\frac{36}{m}$

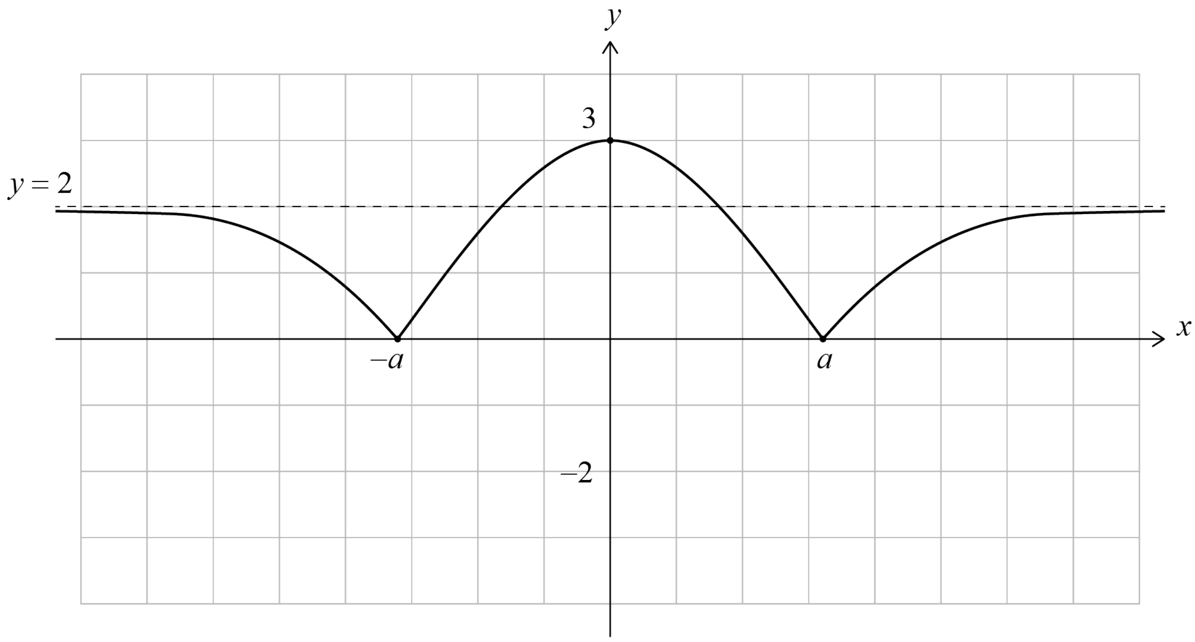
$m=9$ ***(A1)***

third root is 4 ***A1***

> **Note:** Award a maximum of ***A0A0(M1)(M1)(A1)A1*** for a final answer $P(z)=(z-3\mathrm{i})(z+3\mathrm{i})(4-z)$ seen or stating all three correct factors with no evidence of roots throughout their working.

***[6 marks]***

**8.** (a) attempt to reflect $f$ in the $x$ OR $y$ axis ***(M1)***




***A1A1A1***

**Note:** For a curve with an approximately correct shaped right-hand branch, award:
***A1*** for correct asymptotic behaviour at $y = 2$ (either side)
***A1*** for correctly reflected RHS of the graph in the *y*-axis with smooth maximum at (0, 3).
***A1*** for labelled *x*-intercept at $(-a, 0)$ and labelled asymptote at $y = 2$ with sharp points (cusps) at the x-intercepts.

***[4 marks]***

(b) $k = 0$ ***A1***

$4 \le k < 9$ ***A2***

**Note:** If final answer incorrect, award ***A1*** for critical values 4 and 9 seen anywhere.
Exception to FT:
Award a maximum of ***A0A2FT*** if their graph from (a) is not symmetric about the *y*-axis.

***[3 marks]***

***Total [7 marks]***

**9.** **METHOD 1 (subtracting volumes)**

radius of cylinder, $R$ is $\sqrt{r^2 - \frac{h^2}{4}}$ OR $R^2 = r^2 - \frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits 0 and $\frac{h}{2}$ OR $-\frac{h}{2}$ and $\frac{h}{2}$ (seen anywhere) ***(A1)***

**EITHER**

volume of part sphere $= \pi\int\left(r^2 - y^2\right)\mathrm{d}y$

correct integration ***A1***

$r^2 y - \frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{r^2 h}{2} - \frac{h^3}{24}$

recognition that the volume of the ring is $\pi\int\left(r^2 - y^2\right)\mathrm{d}y - \pi R^2 h$ where $R \neq r$ ***(M1)***

$\pi\int\left(r^2 - y^2\right)\mathrm{d}y - \pi\left(r^2 - \frac{h^2}{4}\right)h$ (or equivalent)

correct equation ***(A1)***

$2\pi\left(\frac{r^2 h}{2} - \frac{h^3}{24}\right) - \pi r^2 h + \frac{\pi h^3}{4} = \pi$ OR $\frac{h^3}{4} - \frac{h^3}{12} = 1$ (or equivalent)

**OR**

recognition that the volume of the ring is $\pi\int\left(\left(r^2 - y^2\right) - \left(r^2 - \frac{h^2}{4}\right)\right)\mathrm{d}y$ (or equivalent) ***(M1)***

correct integration ***A1***

$\frac{h^2}{4}y - \frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{h^3}{8} - \frac{h^3}{24}$

correct equation ***(A1)***

$2\pi\left(\frac{h^3}{8} - \frac{h^3}{24}\right) = \pi$ OR $2\left(\frac{h^3}{8} - \frac{h^3}{24}\right) = 1$ (or equivalent)

**THEN**

$h = \sqrt[3]{6}$ ***A1***

***[7 marks]***

*continued…*

*Question 9 continued*

**METHOD 2 (volume of cylindrical hole)**

radius of cylinder, $R$ is $\sqrt{r^2 - \frac{h^2}{4}}$ OR $R^2 = r^2 - \frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits $\frac{h}{2}$ and $r$ (seen anywhere) ***(A1)***

volume of part sphere $= \pi\int\left(r^2 - y^2\right)\mathrm{d}y$

correct integration ***A1***

$r^2 y - \frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{2r^3}{3} - \frac{r^2 h}{2} + \frac{h^3}{24}$

recognition that the volume of the cylindrical hole is $\pi\int\left(r^2 - y^2\right)\mathrm{d}y + \pi R^2 h$ where $R \neq r$ ***(M1)***

$\pi\int\left(r^2 - y^2\right)\mathrm{d}y + \pi\left(r^2 - \frac{h^2}{4}\right)h \ \left(= \frac{4}{3}\pi r^3 - \pi\right)$(or equivalent)

correct equation ***(A1)***

$2\pi\left(\frac{2r^3}{3} - \frac{r^2 h}{2} + \frac{h^3}{24}\right) + \pi r^2 h - \frac{\pi h^3}{4} = \frac{4}{3}\pi r^3 - \pi$ OR $\frac{h^3}{12} - \frac{h^3}{4} = -1$ (or equivalent)

$h = \sqrt[3]{6}$ ***A1***

***[7 marks]***

*continued…*

*Question 9 continued*

**METHOD 3 (shells)**

radius of cylinder, $R$ is $\sqrt{r^2 - \frac{h^2}{4}}$ OR $R^2 = r^2 - \frac{h^2}{4}$ (seen anywhere) ***(A1)***

attempt to use shells method ***(M1)***

$2\pi\int x\sqrt{r^2 - x^2}\,\mathrm{d}x$

correct limits *r* and $\sqrt{r^2 - \frac{h^2}{4}}$ (seen anywhere) ***(A1)***

correct integration ***A1***

$-\frac{1}{3}\left(r^2 - x^2\right)^{\frac{3}{2}}$

attempt to substitute their limits into their integrated expression ***(M1)***

$-\frac{1}{3}\left(0 - \left(r^2 - \left(r^2 - \frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)$

correct equation ***(A1)***

$2 \times \frac{-2\pi}{3}\left(0 - \left(r^2 - \left(r^2 - \frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right) = \pi$ OR $2\left(\frac{2\pi}{3} \times \frac{h^3}{8}\right) = \pi$

$h = \sqrt[3]{6}$ ***A1***

***[7 marks]***

## Section B

**10.** (a) (i) recognition that $n = 5$ ***(M1)***

$S_5 = 45$ ***A1***

(ii) **METHOD 1**

recognition that $S_5 + u_6 = S_6$ ***(M1)***

$u_6 = 15$ ***A1***

**METHOD 2**

recognition that $60 = \frac{6}{2}(S_1 + u_6)$ ***(M1)***

$60 = 3(5 + u_6)$

$u_6 = 15$ ***A1***

**METHOD 3**

substituting their $u_1$ and $d$ values into $u_1 + (n-1)d$ ***(M1)***

$u_6 = 15$ ***A1***

***[4 marks]***

(b) recognition that $u_1 = S_1$ (may be seen in (a)) OR substituting their $u_6$ into $S_6$ ***(M1)***

OR equations for $S_5$ and $S_6$ in terms of $u_1$ and $d$

$1 + 4$ OR $60 = \frac{6}{2}(u_1 + 15)$

$u_1 = 5$ ***A1***

***[2 marks]***

*continued…*

*Question 10 continued*

(c) **EITHER**

valid attempt to find $d$ (may be seen in (a) or (b)) ***(M1)***

$d = 2$ ***(A1)***

**OR**

valid attempt to find $S_n - S_{n-1}$ ***(M1)***

$n^2 + 4n - \left(n^2 - 2n + 1 + 4n - 4\right)$ ***(A1)***

**OR**

equating $n^2 + 4n = \frac{n}{2}\left(5 + u_n\right)$ ***(M1)***

$2n + 8 = 5 + u_n$ (or equivalent) ***(A1)***

**THEN**

$u_n = 5 + 2(n-1)$ OR $u_n = 2n + 3$ ***A1***

***[3 marks]***

(d) recognition that $v_2 r^2 = v_4$ OR $\left(v_3\right)^2 = v_2 \times v_4$ ***(M1)***

$r^2 = 3$ OR $v_3 = (\pm)5\sqrt{3}$ ***(A1)***

$r = \pm\sqrt{3}$ ***A1***

**Note:** If no working shown, award ***M1A1A0*** for $\sqrt{3}$ .

***[3 marks]***

(e) recognition that $r$ is negative ***(M1)***

$v_5 = -15\sqrt{3} \quad \left(= -\frac{45}{\sqrt{3}}\right)$ ***A1***

***[2 marks]***

***Total [14 marks]***

**11.** (a) $L = \text{AC} + \text{CB}$

$\dfrac{\left(\frac{3}{4}\right)}{\text{AC}} = \cos\alpha \left( \Rightarrow \text{AC} = \dfrac{\frac{3}{4}}{\cos\alpha} \Rightarrow \text{AC} = \frac{3}{4}\sec\alpha \right)$ ***A1***

$\dfrac{6}{\text{CB}} = \sin\alpha \left( \Rightarrow \text{CB} = \dfrac{6}{\sin\alpha} \Rightarrow \text{CB} = 6\operatorname{cosec}\alpha \right)$ ***A1***

so $L = \frac{3}{4}\sec\alpha + 6\operatorname{cosec}\alpha$ ***AG***

***[2 marks]***

(b) (i) $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \frac{3}{4}\sec\alpha\tan\alpha - 6\operatorname{cosec}\alpha\cot\alpha$ ***A1***

(ii) attempt to write $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha}$ in terms of $\sin\alpha, \cos\alpha$ or $\tan\alpha$ (may be seen in (i)) ***(M1)***

$\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{\frac{3}{4}\sin\alpha}{\cos^2\alpha} - \dfrac{6\cos\alpha}{\sin^2\alpha}$ OR $\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = \dfrac{\frac{3}{4}\tan\alpha}{\cos\alpha} - \dfrac{6}{\sin\alpha\tan\alpha} \left( = \dfrac{\frac{3}{4}\tan^3\alpha - 6}{\cos\alpha\tan^2\alpha} \right)$

$\dfrac{\mathrm{d}L}{\mathrm{d}\alpha} = 0 \Rightarrow \frac{3}{4}\sin^3\alpha - 6\cos^3\alpha = 0$ OR $\frac{3}{4}\tan^3\alpha - 6 = 0$ (or equivalent) ***(A1)***

$\tan^3\alpha = 8$ ***A1***

$\tan\alpha = 2$ ***A1***

$\alpha = \arctan 2$ ***AG***

***[5 marks]***

*continued…*

*Question 11 continued*

(c) (i) attempt to use product rule (at least once) ***(M1)***

$$\frac{d^2L}{d\alpha^2} = \frac{3}{4}\sec\alpha\tan\alpha\tan\alpha + \frac{3}{4}\sec\alpha\sec^2\alpha$$

$+6\text{cosec}\,\alpha\cot\alpha\cot\alpha + 6\text{cosec}\,\alpha\,\text{cosec}^2\alpha$ (or equivalent) ***A1A1***

**Note:** Award ***A1*** for $\frac{3}{4}\sec\alpha\tan\alpha\tan\alpha + \frac{3}{4}\sec\alpha\sec^2\alpha$ and ***A1*** for $+6\text{cosec}\,\alpha\cot\alpha\cot\alpha + 6\text{cosec}\,\alpha\,\text{cosec}^2\alpha$. Allow unsimplified correct answer.

$$\left(\frac{d^2L}{d\alpha^2} = \frac{3}{4}\sec\alpha\tan^2\alpha + \frac{3}{4}\sec^3\alpha + 6\text{cosec}\,\alpha\cot^2\alpha + 6\text{cosec}^3\alpha\right)$$

(ii) attempt to find a ratio other than $\tan\alpha$ using an appropriate trigonometric identity OR a right triangle with at least two side lengths seen ***(M1)***

**Note:** Award ***M0*** for $\alpha = \arctan 2$ substituted into their $\frac{d^2L}{d\alpha^2}$ with no further progress.

one correct ratio ***(A1)***

$\sec\alpha = \sqrt{5}$ OR $\text{cosec}\,\alpha = \frac{\sqrt{5}}{2}$ OR $\cot\alpha = \frac{1}{2}$ OR $\cos\alpha = \frac{1}{\sqrt{5}}$ OR $\sin\alpha = \frac{2}{\sqrt{5}}$

**Note:** ***M1A1*** may be seen in part (d).

$$\frac{3}{4}\left(\sqrt{5}\right)\left(2^2\right) + \frac{3}{4}\left(\sqrt{5}\right)^3 + 6\left(\frac{\sqrt{5}}{2}\right)\left(\frac{1}{2}\right)^2 + 6\left(\frac{\sqrt{5}}{2}\right)^3$$ (or equivalent) ***A2***

$$\frac{12\sqrt{5}}{4} + \frac{15\sqrt{5}}{4} + \frac{3\sqrt{5}}{4} + \frac{15\sqrt{5}}{4}$$

**Note:** Award ***A1*** for only two or three correct terms.
Award a maximum of ***(M1)(A1)A1*** on ***FT*** from c(i).

$$\frac{d^2L}{d\alpha^2} = \frac{45}{4}\sqrt{5}$$ ***AG***

***[7 marks]***

*continued…*

*Question 11 continued*

(d) (i) $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} > 0$ OR concave up (or equivalent) ***R1***

(and $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, when $\alpha = \arctan 2$, hence $L$ is a minimum)

(ii) $(L_{\min} =)\frac{3}{4}(\sqrt{5}) + 6\left(\frac{\sqrt{5}}{2}\right)$ ***(A1)***

$= \frac{15\sqrt{5}}{4}$ ***A1***

***[3 marks]***

(e) $(11.25 =)\frac{15\sqrt{9}}{4} > \frac{15\sqrt{5}}{4}$ (or equivalent comparative reasoning) ***R1***

the pole cannot be carried (horizontally from the passageway into the room) ***A1***

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| **Note:** Do not award ***R0A1***. |

***[2 marks]***

***Total [19 marks]***

**12.** (a) $2t + 1 \times 0 + 0 \times (3 + t) \quad (= 2t)$ (seen anywhere) ***(A1)***

one correct magnitude $\sqrt{1^2 + 1^2 + 0^2}, \sqrt{(2t)^2 + (3 + t)^2}$ ***(A1)***

correct substitution of their magnitudes and scalar product ***M1***

$2t = \sqrt{2} \times \sqrt{(2t)^2 + (3 + t)^2} \times \cos\frac{\pi}{3}$ OR $\cos\frac{\pi}{3} = \frac{2t}{\sqrt{2} \times \sqrt{5t^2 + 6t + 9}}$

$4t = \sqrt{2\left(4t^2 + 9 + 6t + t^2\right)}$ OR $\frac{1}{2} = \frac{2t}{\sqrt{2\left(5t^2 + 6t + 9\right)}}$ (or equivalent) ***A1***

$4t = \sqrt{10t^2 + 12t + 18}$ ***AG***

***[4 marks]***

(b) correct quadratic equation ***A1***

$16t^2 = 10t^2 + 12t + 18$, $6t^2 - 12t - 18 = 0$ , $t^2 - 2t - 3 = 0$

valid attempt to solve their quadratic set =0 ***(M1)***

$(t + 1)(t - 3)$ OR $\frac{12 \pm \sqrt{(-12)^2 - 4 \times 6 \times (-18)}}{12}$ OR $(t - 1)^2 - 4$ ***(A1)***

$t = 3$ ***A1***

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| **Note:** Award ***A0*** if additional answer(s) given. |

***[4 marks]***

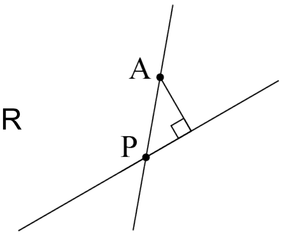
*continued…*

*Question 12 continued*

(c) **METHOD 1**

recognizing shortest distance from A is perpendicular to $L_1$ ***(M1)***

OR




$\sin\frac{\pi}{3} = \frac{x}{\left|\overrightarrow{PA}\right|}$

$\left|\overrightarrow{PA}\right| = \sqrt{6^2+6^2} \quad \left(=\sqrt{72},\ 6\sqrt{2}\right)$ (seen anywhere) ***(A1)***

$\frac{\sqrt{3}}{2} = \frac{x}{\sqrt{72}}$ ***(A1)***

$x = \frac{\sqrt{216}}{2} \quad \left(=\sqrt{54}\ ,\ 3\sqrt{6}\right)$

shortest distance is $\frac{\sqrt{216}}{2}\left(=\sqrt{54}\ ,\ 3\sqrt{6}\right)$ ***A1***

***[4 marks]***

*continued…*

*Question 12 continued*

**METHOD 2**

recognition that the distance required is $\frac{\left|\boldsymbol{v} \times \overrightarrow{\mathrm{PA}}\right|}{|\boldsymbol{v}|}$ ***(M1)***

$= \frac{1}{\sqrt{2}}\left|\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}6\\0\\6\end{pmatrix}\right|$ ***(A1)***

$= \frac{1}{\sqrt{2}}\left|\begin{pmatrix}6\\-6\\-6\end{pmatrix}\right|$ ***(A1)***

shortest distance is $\sqrt{54}\left(=3\sqrt{6}\right)$ ***A1***

***[4 marks]***

*continued…*

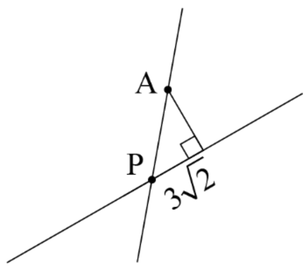
*Question 12 continued*

**METHOD 3**

recognition that the base of the triangle is $\frac{\left|\boldsymbol{v}\cdot\overrightarrow{\mathrm{PA}}\right|}{|\boldsymbol{v}|}$ ***(M1)***

$\frac{1}{\sqrt{2}}\left|\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}6\\0\\6\end{pmatrix}\right|$

$=\frac{6}{\sqrt{2}}\ \left(=3\sqrt{2}\right)$ OR ***(A1)***




$\left|\overrightarrow{\mathrm{PA}}\right|=\sqrt{6^2+6^2}\quad\left(=\sqrt{72},\ 6\sqrt{2}\right)$ (seen anywhere) ***(A1)***

> **Note:** The value of $\left|\overrightarrow{\mathrm{PA}}\right|=\sqrt{6^2+6^2}$ may be seen as part of the working of their shortest distance, $d=\sqrt{\left|\overrightarrow{\mathrm{PA}}\right|^2-b^2}=\sqrt{\left(\sqrt{72}\right)^2-\left(3\sqrt{2}\right)^2}$

shortest distance is $\sqrt{54}\ \left(=3\sqrt{6}\right)$ ***A1***

***[4 marks]***

*continued…*

*Question 12 continued*

**METHOD 4**

Let B be a general point on $L_1$ $(\lambda, 8+\lambda, -3)$ such that AB is perpendicular to $L_1$

attempt to find vector $\vec{AB}$ OR $\left|\vec{AB}\right|$ (the shortest distance from A to $L_1$) ***(M1)***

$$\vec{AB} = \vec{OB} - \vec{OA} = \vec{OP} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} - \vec{OA} \left( = \begin{pmatrix}0\\8\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}6\\8\\3\end{pmatrix} = \vec{AP} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}\right) \quad (\lambda \in \mathbb{R})$$

$\vec{AB} = \begin{pmatrix}-6\\0\\-6\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$ OR $\left|\vec{AB}\right| = \sqrt{(\lambda-6)^2 + (8+\lambda-8)^2 + (-3-3)^2}$ ***A1***

$$\left|\vec{AB}\right| = \sqrt{(\lambda-6)^2 + \lambda^2 + (-6)^2} \left( = \sqrt{2\lambda^2 - 12\lambda + 72}\right)$$

**EITHER**

$\dfrac{d}{d\lambda}\left(\left|\vec{AB}\right|^2\right) = 0 \Rightarrow 4\lambda - 12 = 0 \Rightarrow \lambda = 3$ ***A1***

**OR**

$\left|\vec{AB}\right| = \sqrt{2(\lambda-3)^2 + 54}$ to obtain $\lambda = 3$ ***A1***

**OR**

$\begin{pmatrix}-6+\lambda\\ \lambda\\ -6\end{pmatrix} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} = 0 \Rightarrow -6 + \lambda + \lambda = 0 \Rightarrow \lambda = 3$ ***A1***

**THEN**

shortest distance is $\sqrt{54}$ $\left(= 3\sqrt{6}\right)$ ***A1***

***[4 marks]***

*continued…*

*Question 12 continued*

(d) attempt to find the vector product of two direction vectors ***(M1)***

$$\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}6\\0\\6\end{pmatrix}$$

$\boldsymbol{n}=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ (or any scalar multiple of this) (accept $\boldsymbol{n}=<1,-1,-1>$ or equivalent) ***A1***

| **Note:** Award ***A0*** for a final answer given in coordinate form. |
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***[2 marks]***

*continued…*

*Question 12 continued*

(e) substituting their $x$ into volume formula and equating ***(M1)***

$\frac{1}{3}\pi\left(3\sqrt{6}\right)^2 h = 90\sqrt{3}\pi$

$h = 5\sqrt{3}$ (seen anywhere) ***A1***

recognition that the position vector of vertex is given by $\overrightarrow{OA} + \mu\boldsymbol{n}$ OR $\overrightarrow{OA} + h \times \hat{\boldsymbol{n}}$ ***(M1)***

$\begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ OR $(6+\mu, 8-\mu, 3-\mu)$

**EITHER**

recognition that $\mu|\boldsymbol{n}| = h$ (where $\mu$ is a parameter) ***(M1)***

$\mu|\boldsymbol{n}| = 5\sqrt{3}$ OR $\sqrt{\mu^2 + (-\mu)^2 + (-\mu)^2} = 5\sqrt{3}$ OR $3\mu^2 = 75$ $\left(\Rightarrow \sqrt{3}\mu = 5\sqrt{3}\right)$

$\mu = \pm 5$ (accept $\mu = 5$) ***(A1)***

**OR**

attempt to find cone's height vector $h \times \hat{\boldsymbol{n}}$ ***(M1)***

$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ ***(A1)***

$5\sqrt{3} \times \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

**THEN**

$= \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} \pm 5\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \left( = \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} \pm \begin{pmatrix} 5 \\ -5 \\ -5 \end{pmatrix} \right)$

vertex = $(11, 3, -2)$ and $(1, 13, 8)$ (accept position vectors) ***A1A1***

**Note:** Award a maximum of ***(M0)A0(M1)(M1)(A1)A1A1FT*** for $\left(\frac{39}{4}, \frac{17}{4}, -\frac{3}{4}\right)$ and $\left(\frac{9}{4}, \frac{47}{4}, \frac{27}{4}\right)$ obtained using $x = \left|\overrightarrow{PA}\right|$ from part (c).

***[7 marks]***

***Total [21 marks]***